NON-NEWTONIAN OSCILLATORY LAYER FLOW, APPROXIMATE SOLUTION

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The problem of the oscillatory flow of pseudoplastic liquid in vicinity of the infinitely long horizontal plane is formulated in stresses. For Re $\rightarrow \infty$ *i.e.* for conditions of oscillatory boundary layer the problem is solved approximately by the Galerkin method.

A detailed theoretical study of oscillatory flows in non-Newtonian liquids is significant first of all in relation to the possibility of mechanical liquefying of pseudoplastic materials by vibration of walls[1]. Analytical approximations[2] which have been proposed and which are based on motion equations given in velocities have not led to reasonable estimates of the degree of liquefying. The corresponding limiting problem is here formulated in stresses which, in simple analytical approximations of the stress field, leads to considerably better estimates of the degree of liquefying than the earlier published approximations of the velocity field[2].

FORMULATION OF THE PROBLEM IN STRESSES

The motion of the film of liquid material of the thickness h is considered situated on the horizontal flat plate in unidirectional oscillatory motion in its own plane. The Cartesian coordinates (x, y, z) are selected so that x is changing in the direction perpendicular to the plane and z is changing in the direction parallel with the direction of the plane motion. The basical spacial symmetries of the problem can be expressed by the relations for Cartesian coordinates of velocity and stress

$$v_x = v_y = 0\,, \quad v_z = v(t, x) \tag{1a}$$

$$\tau_{xy} = \tau_{yz} = 0\,, \quad \tau_{zx} = \tau(t, x)\,, \tag{1b}$$

where t is the time variable.

The differential momentum balance in the direction of action of the plane can be expressed by the relation

$$\varrho\, \partial_t v = \partial_x \tau\,. \tag{2}$$

Only purely viscous materials are considered with the viscosity function

$$\gamma = \tilde{\gamma}[\tau], \quad \tilde{\gamma}[-\tau] = -\tilde{\gamma}[\tau]. \qquad (3a,b)$$

where γ is the shear velocity

$$\gamma = \partial_x v. \qquad (4)$$

The boundary conditions of the problem at the assumption that there is no slip between the material and the oscillating wall and that on the free surface of the material no net forces act can be formulated by equations

$$v(t, x) = v_0(t); \quad x = 0 \qquad (5)$$

$$\tau(t, x) = 0; \qquad x = h. \qquad (6)$$

With respect to Eq. (2) the boundary condition (5) can be substituted by the condition in stresses

$$\partial_x \tau = \varrho\, \partial_t\, v_0(t); \quad x = 0 \qquad (7)$$

and the condition (6) can be, on the contrary, expressed in velocities

$$\partial_x v = 0; \quad x = h. \qquad (8)$$

The boundary conditions, in the case when $v_0(t)$ reflects periodical asymmetry

$$v_0(t + \pi/\omega) = -v_0(t), \qquad (9)$$

can be supplemented by similar conditions for the field $v(t, x)$ and $\tau(t, x)$

$$v(t + \pi/\omega, x) = -v(t, x) \qquad (10a)$$

$$\tau(t + \pi/\omega, x) = -\tau(t, x). \qquad (10b)$$

The equation of motion can be also expressed either in velocities

$$\varrho\, \partial_t v = \partial_x\{\tilde{\tau}[\partial_x v]\}, \qquad (11)$$

the function $\tau = \tilde{\tau}(\gamma)$ being inversion to the function $\gamma = \tilde{\gamma}[\tau]$, or in stresses

$$\varrho\, \partial_t\{\tilde{\gamma}[\tau]\} = \partial^2_{xx}\tau. \qquad (12)$$

Thus it depends only on our choice if the considered physical problem is solved in stresses, Eqs (*6*), (*7*), (*10b*) and (*12*) or in velocities, Eqs (*5*), (*8*), (*10a*) and (*11*). Both fields are related by relation

$$v(t, x) = v_0(t) + \int_0^x \tilde{\gamma}[\tau(t, \xi)] \, d\xi \qquad (13a)$$

or

$$\tau = \tilde{\tau}[\partial_x v] \, . \qquad (13b)$$

It is more usual to formulate and solve the problems of the considered type in velocities, see literature survey in our recent publication[1]. Nevertheless, the solution of the problem in stresses has at least two advantages. First, at the calculation of the velocity field from the stress field according to Eq. (*13a*) only integration is needed, while the inversion operation according to (*13b*) requires derivation of the velocity field. At second, the boundary condition (*7*) for the stress field directly requires the differential momentum balance on the wall. This has a certain significance in approximative solutions of the problem by the Galerkin method.

Let some field $F(t, x)$ be defined by the partial differential equation

$$\mathcal{D}\{F\} = 0 \, , \quad (t, x) \in A \qquad (14)$$

and by the corresponding boundary conditions. The field F is approximated by some trial function $f(t, x; \alpha_1, \ldots, \alpha_N, \ldots)$ with certain, in general unrestricted, number of adjustable parameters $(\alpha_1, \ldots, \alpha_N, \ldots)$. Let the function f be introduced so that for the arbitrarily and independently chosen parameters (α_k) it satisfies all boundary conditions of the problem. The Galerkin method of determination of parameters $(\alpha_1, \ldots, \alpha_N, \ldots)$ is based on solution of the system of algebraic equations of the type

$$\iint_A \partial_i f \, \mathcal{D}\{f\} \, dt \, dx = 0 \, ; \quad i = 1, \ldots, N, \ldots \qquad (15)$$

$$\partial_i f = \partial f(t, x; \alpha_1, \ldots, \alpha_i, \ldots, \alpha_N, \ldots)/\partial \alpha_i \, . \qquad (16)$$

We can see that for the considered problem of periodical oscillations more *a priori* dynamic requirements on the system of trial functions are included in formulations in stresses than in the formulation in velocities. Beside the integral conditions (*15*) the stress field is bound also by the dynamic condition $\mathcal{D}\{f\} = 0$ on the wall which is expressed by the relation (*7*). The kinematic condition (*5*) is of significance in the case of calculation of the velocity field according to (*13a*).

APPROXIMATIVE SOLUTION for Re $\to \infty$

Here, only the special case of the above formulated problem is considered. Harmonic oscillations of the plane are considered

$$v_0 = a\omega \cos(\omega t) \tag{17}$$

with the power-law viscosity function

$$\tilde{\gamma}[\tau] = [\tau/K]^{1/n} = -[-\tau/K]^{1/n} . \tag{18}$$

If the parameters a, ω, ϱ, K are used for introduction of the normalized variables

$$Y = x/L_B\,; \quad T = t\omega\,; \quad V = v/(a\omega)\,; \quad S = \tau/\tau_B\,, \tag{19}$$

where

$$\tau_B = (K(\varrho a^2\omega^3)^n)^{1/(1+n)}$$

$$L_B = (K\varrho^{-1}a^{-1+n}\omega^{-2+n})^{1/(1+n)} \tag{20}$$

are considered to be the characteristic stress and characteristic length, the normalised formulation of the boundary problem is reached in stresses

$$\mathscr{D}\{S\} \equiv \partial_T[S]^{1/n} - \partial^2_{YY}S = 0 \tag{21}$$

$$S(T + \pi, Y) = -S(T, Y) \tag{22}$$

$$\partial_Y S = -\sin T\,; \quad Y = 0 \tag{23}$$

$$S = 0\,; \quad Y = \mathrm{Re}^{1/(1+n)} . \tag{24}$$

The only macroscopic parameter of the problem is the Reynolds number in the form

$$\mathrm{Re} = h^{1+n}a^{1-n}\omega^{2-n}\varrho K^{-1} . \tag{25}$$

In the following part, only the asymptotic case

$$\mathrm{Re} \to \infty\,, \tag{26}$$

is considered, corresponding to the cases

$$h \to \infty\,, \quad a \to \infty\,, \quad \text{resp.} \quad \omega \to \infty\,.$$

The approximative solution is supposed, with regard to the structure of the exact solution[2] for $n = 1$ in the form

$$S(T, Y; A, B) = \Delta^{-2} \exp(-AY)(AS^* - BC^*), \tag{27}$$

$$S^* = \sin(T - BY) \tag{28a}$$

$$C^* = \cos(T - BY) \tag{28b}$$

$$\Delta = (A^2 + B^2)^{1/2}, \tag{29}$$

with the pair of adjustable parameters A and B.

Determination of parameters A and B by the Galerkin method leads to the solution of the system of two equations

$$\int_0^{2\pi}\int_0^{\infty} \partial_A S\{\partial_T[S]^{1/n} - \partial^2_{YY} S\} \, dY dT = 0 \tag{30a}$$

$$\int_0^{2\pi}\int_0^{\infty} \partial_B S\{\partial_T[S]^{1/n} - \partial^2_{YY} S\} \, dY dT = 0, \tag{30b}$$

where the trial function S is defined by Eq. (*27*). Substitution of Eq. (*27*) into (*30a,b*) leads to the system of two algebraic equations for A and B with the solution

$$A = B = \frac{1}{\sqrt{2}} \left(\frac{2\Gamma(1/(2n))}{(1 + n)\sqrt{\pi}\,\Gamma(1/(2n) + 3/2)} \right)^{n/(1+n)} \tag{31}$$

i.e.

$$S(T, Y) \approx 2^{-1/2} A^{-1} \exp(-AY) \sin(T - AY - \pi/4). \tag{32}$$

This solution is identical for $n = 1$ with the exact solution of the considered problem.

For application of the theory of periodical shears[1] in the region of large Re values the function $\varkappa(n)$

$$\varkappa(n) = \frac{1 + 2n}{n^2 2\pi} \int_0^{\infty}\int_0^{2\pi} |S(T, Y)|^{1/n - 1} \, dY dT, \tag{33}$$

is significant, giving the effect of periodical oscillations of the effective viscosity in the oscillating boundary layer. If the stress field is approximated by the relation (*32*), the corresponding approximation of the function $\varkappa(n)$ can be found in the form

$$\varkappa(n) \approx \frac{(1 + 2n)(1 + n)}{n(1 - n)} A^2 \tag{34}$$

with asymptotic approximations

$$\varkappa(n) \approx \begin{cases} \dfrac{3}{1 - n} - 2{\cdot}58\,; & (n \to 1_-) \\[2ex] \dfrac{1 + 4n}{2n} (2{\cdot}16n)^{3n}\,; & (n \to 0_+)\,. \end{cases} \tag{35a,b}$$

DISCUSSION

The boundary problem represented by the system of Eqs (*21*) to (*23*) has for $1/n \gg \gg 1$ an essential nonlinearity due to which the possibility of exact analytical solution is very improbable. The proposed numerical solutions of this problem[3-5] by the mesh method have in the region of intermediate Re a considerable inaccuracy, characterized by a 10–20% error in integral momentum and mechanical energy balances. Moreover in the region of large Re the solution by the mesh method fails, as for $T =$ const. the velocity profile has an alternating shape with extremes and inflexes whose number increases in direct proportion with $\mathrm{Re}^{1/(1+n)}$. This was the reason why we have attempted an analytical approximation of the velocity field[2] and also for the presented attempt for approximation of the stress field.

The approximation of the stress field according to Eqs (*31*) and (*32*) has in comparison with the analytical approximation[2] certain advantage partly in that it satisfies the boundary condition of the wall Eq. (*23*) and that it leads for $n \to 0$ to finite values of stresses on the walls. Another advantage of the new approximation is the considerably better estimate of the size of liquefying in the regime of oscillating boundary layer[1].

In a number of other considerations *e.g.* at comparison with the exact solution for ideal plastic material[6] the structure of the two-parameter trial function (*27*) does not seem quite suitable and the question of improvement of the approximation by a suitable selection of the multiparameter trial function will be subject of our next studies.

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